

On Perturbations of a Quintom Bounce

Yi-Fu Cai^{1*}, Taotao Qiu^{1†}, Robert Brandenberger^{2,3‡}, Yun-Song Piao^{4§}, Xinmin Zhang^{1,5¶}

¹ *Institute of High Energy Physics, Chinese Academy of Sciences, P.O. Box 918-4, Beijing 100049, P. R. China*

² *Department of Physics, McGill University, Montreal, QC, H3A 2T8, Canada*

³ *Kavli Institute for Theoretical Physics, Zhong Guan Cun East Street 55, Beijing 100080, P.R. China*

⁴ *College of Physical Sciences, Graduate School of Chinese Academy of Sciences, Beijing 100049, P. R. China*

⁵ *Theoretical Physics Center for Science Facilities (TPCSF), CAS, P. R. China*

Abstract

A Quintom universe with an equation-of-state crossing the cosmological constant boundary can provide a bouncing solution dubbed the Quintom Bounce and thus resolve the Big Bang singularity. In this paper, we investigate the cosmological perturbations of the Quintom Bounce both analytically and numerically. We find that the fluctuations in the dominant mode in the post-bounce expanding phase couple to the growing mode of the perturbations in the pre-bounce contracting phase.

*caiyf@mail.ihep.ac.cn

†qiutt@mail.ihep.ac.cn

‡rhh@hep.physics.mcgill.ca

§yspiao@gucas.ac.cn

¶xmzhang@mail.ihep.ac.cn

1 Introduction

For years, it has been suggested that bouncing universe scenarios could provide a solution to the problem of the initial singularity in the evolution of the cosmos, a problem which afflicts both Standard Big Bang cosmology and inflationary universe models [1, 2, 3, 4]^{||}. In the literature there have been a lot of efforts towards constructing bouncing universes, for instance the Pre-Big-Bang scenario [7] or the cyclic/Ekpyrotic scenario [8]. Also, in loop quantum cosmology the evolution might be singularity-free [9]. In Refs. [10, 11, 12] and [13, 14] models were considered in which the gravitational action is modified by adding high order terms. Since, in general, in these models the equations for cosmological perturbations are very involved, it is difficult to make definitive conclusions about the evolution of fluctuations, and the knowledge of the evolution is crucial in order to compare the predictions of the model with observations.

In some previous work [15], a subset of the present authors considered a bouncing universe model (“Quintom Bounce”) obtained within the standard 4-dimensional Friedmann-Robertson-Walker (FRW) framework, but making use of Quintom matter [16]. In this model, it can be shown in [15] that the null energy condition (NEC) is violated for a short period around the bounce point. Moreover, after the bounce the equation-of-state (EoS) of the matter content w in our model is able to transit from $w < -1$ to $w > -1$ and then connect with normal expanding history.

A Quintom model [16] was initially proposed to obtain a model of dark energy with an EoS parameter w which satisfies $w > -1$ in the past and $w < -1$ at present. This model is mildly favored by the current observational data fitting [17]. The Quintom model is a dynamical scenario of dark energy with a salient feature that its EoS can smoothly cross over the cosmological constant barrier $w = -1$. It is not easy to construct a consistent Quintom model. A no-go theorem proven in Ref. [18] (also see Refs. [16, 19, 20, 21, 22, 23, 24]) forbids a traditional scalar field model with Lagrangian of the general form $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi \partial^\mu \phi)$ from having its EoS cross the cosmological constant boundary. Therefore, it is necessary to add extra degrees of freedom with un-conventional features to the conventional single field theory if we expect to realize a viable Quintom model in the framework of Einstein’s gravity theory. The simplest Quintom model involves two scalars with one being Quintessence-like and another Phantom-like [16, 25]. This model has been studied in detail in [26]. In recent years there have been a lot of theoretical studies of Quintom models. For example, motivated from string theory, the authors of Ref. [27] realized a Quintom scenario by considering the non-perturbative effects of a DBI model. Moreover, there are models which involve a single scalar with higher derivative terms in the action [28], models with vector field [29], making use of an extended theory of gravity [30], of ideas from non-local string field theory [31], and others (see e.g. [32, 33, 34]). Because of its brand new properties, Quintom models give many unexpected predictions. Most important for the subject of the present paper is that a universe dominated by Quintom matter can provide a bouncing cosmology which allows us to avoid the problem of the initial singularity. This scenario, developed in [15], is called the Quintom Bounce. In this scenario, the EoS crosses the cosmological constant boundary twice around the bounce point.

In order to be able to compare the predictions of a Quintom bounce with observations, it is crucial to investigate the evolution of cosmological perturbations. In singular bouncing models the evolution of fluctuations is not under control classically. The fluctuations diverge at the bounce point, firstly invalidating the applicability of the theory of linear cosmological perturbations, and secondly making it

^{||}An initial singularity occurs in inflationary models if scalar fields are used to generate the accelerated expansion of space [5]. If inflation arises as a consequence of modifications of the gravitational part of the action, as in Starobinsky’s original proposal [6], it is possible that the evolution is singularity-free.

impossible to reliably connect the fluctuations in the contracting phase with those in the expanding phase. In a Quintom bounce model the evolution of fluctuations is well-behaved. Our first main result is that cosmological perturbations can evolve smoothly from the contracting to the expanding phase. We show that the commonly used variable Φ , the metric perturbation in longitudinal gauge, passes smoothly both through the bounce point and through the points where the cosmological constant boundary of the EoS is crossed. Interestingly, we find that our result for the transfer of fluctuations through the bounce possesses features found both in some previous analyses of fluctuations in non-singular cosmologies, and features of the evolution in singular bouncing models. For small comoving wave numbers one finds an evolution similar to what occurs in singular bouncing cosmologies (the growing mode of Φ in the contracting phase couples only to the decaying mode in the expanding phase), whereas the transfer of the perturbations is very different in the region of large comoving wave numbers (the growing mode in the contracting phase couples to the dominant mode in the phase of expansion). The physical reason is easy to see: small k modes enter the sub-Hubble region very close to the bounce point and have no time to complete a single oscillation, whereas large k modes undergo many oscillations which are in the bounce phase at sub-Hubble scales. Thus, whereas the large k modes feel the resolution of the singularity, the small k modes evolve as if there still were an abrupt transition between the contracting to the expanding phase. We will follow the fluctuations both numerically and analytically. In the analytical analysis, we apply matching conditions away from the bounce point, in regions where both the background and the perturbations can be matched consistently via the Deruelle-Mukhanov [35] (initially derived in [36]) conditions **.

This paper is organized as follows. In Section 2, we review the equations of motion of double-field Quintom and derive the perturbation equations for this model. In Section 3, we study the behavior of a specific Quintom Bounce model systematically. In Subsection 3.1, we determine the complete evolution of the background using numerical integration. In Subsection 3.2, we investigate the perturbations, first analytically by solving the equations in separate periods by means of approximations valid during those periods, and matching the solutions at the transition points, and finally numerically. The numerical results support our analytical calculations consistently for all times. Section 4 contains discussion and conclusions.

2 Review of the Quintom Bounce in a Double-Field Model

2.1 Equations of Motion of Double-Field Quintom

To start, we take a Quintom model consisting of two fields with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - V(\phi) - W(\psi) , \quad (1)$$

where the signature of the metric is $(+, -, -, -)$. Here the field ϕ has a canonical kinetic term, but ψ has a kinetic term with the opposite sign and thus plays a role of a ghost field. In the framework of a flat Friedmann-Robertson-Walker (FRW) universe, the metric is given by $ds^2 = dt^2 - a^2(t)dx^i dx^i$. By varying the corresponding matter action, we easily obtain the following expressions for the energy density ρ and pressure p of this model,

$$\rho = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\psi}^2 + V(\phi) + W(\psi) , \quad p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\psi}^2 - V(\phi) - W(\psi) , \quad (2)$$

**For a discussion of the dangers in applying the matching conditions at the transition point between contraction and expansion see [37].

and the background equations of motion are given by

$$H^2 = \frac{8\pi G}{3}[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\psi}^2 + V + W] , \quad (3)$$

$$\dot{H} = -4\pi G(\dot{\phi}^2 - \dot{\psi}^2) , \quad (4)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 , \quad (5)$$

$$\ddot{\psi} + 3H\dot{\psi} - W_{,\psi} = 0 , \quad (6)$$

where $a(t)$ is the scale factor, $H = \frac{\dot{a}}{a}$ is the Hubble parameter and the dot denotes the derivative with respect to the cosmic time t .

It is well known that this model can realize the EoS w of a universe which crosses the cosmological constant boundary. Namely, the expression for w is given by

$$w = -1 + \frac{\dot{\phi}^2 - \dot{\psi}^2}{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\psi}^2 + V(\phi) + W(\psi)} . \quad (7)$$

If $\dot{\phi}^2 - \dot{\psi}^2$ is positive, w is larger than -1 and the universe lies in a Quintessence-like phase; if $\dot{\phi}^2 - \dot{\psi}^2$ is negative, w is less than -1 and correspondingly the universe is Phantom-like. In recent work, it was found that the Quintom model can provide a bouncing universe scenario[15]. If we work in the context of a spatially flat four-dimensional background FRW metric, there must be a period when the Null Energy Condition (NEC) is violated if we are to obtain a smooth transition from a contracting universe into an expanding phase. In this case, we need a kind of matter which admits an EoS parameter which is less than -1 , but only around the bounce. Neither regular nor Phantom matter alone can achieve a transition in the EoS parameter through the cosmological constant boundary. Therefore, a Quintom model is the only possible solution to resolve this difficulty.

2.2 Equations of Motion of Perturbations

Now let us consider linear perturbations of the metric. The longitudinal (conformal-Newtonian) gauge metric perturbations are given by

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)dx^i dx^i] , \quad (8)$$

where we introduced the comoving time η defined by $d\eta = dt/a$ for convenience. The fields Φ and Ψ depend on space and time and contain the information about the scalar metric fluctuations, the degrees of freedom in the metric which couple to matter fluctuations (see e.g. [38] for a comprehensive survey of the theory of cosmological perturbations and [39] for a recent overview). We will not discuss vector fluctuations since they are not induced by scalar field matter. To linear order, tensor metric fluctuations (gravitational waves) do not couple to matter and hence we will not discuss them here. By expanding the Einstein equations to first order, we obtain the following perturbation equations

$$\nabla^2 \Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho , \quad (9)$$

$$\Psi' + \mathcal{H}\Phi = -4\pi G a \delta q , \quad (10)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p , \quad (11)$$

where $\mathcal{H} = da/ad\eta$ is the comoving Hubble parameter and the prime denotes the derivative with respect to the comoving time. Here $\delta\rho$, δq and δp represent the perturbations of density, momentum and pressure, respectively. The first equation is the perturbed ‘00’ equation, the second the perturbed ‘0i’ equation, and

the third is the diagonal ‘ ii ’ equation. For scalar field matter there is no anisotropic stress to linear order in the matter fluctuations, and hence it follows from the off-diagonal ‘ ij ’ perturbed Einstein equations that $\Phi = \Psi$.

From the conservation of the stress tensor, we also obtain the analog of the continuity equation for the matter fluctuations:

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta p) - \frac{1}{a}\nabla^2\delta q = 3(\rho + p)\Phi' . \quad (12)$$

In the double-field Quintom model, the scalar fields are perturbed: $\phi \rightarrow \phi + \delta\phi$ and $\psi \rightarrow \psi + \delta\psi$. To linear order in the scalar field fluctuations $\delta\phi$ and $\delta\psi$, the perturbations of energy, momentum and pressure can be expressed by

$$\delta\rho = \frac{1}{a^2}\phi'(\delta\phi' - \phi'\Phi) + V_{,\phi}\delta\phi - \frac{1}{a^2}\psi'(\delta\psi' - \psi'\Phi) + W_{,\psi}\delta\psi , \quad (13)$$

$$\delta q = \frac{1}{a}[-\phi'\delta\phi + \psi'\delta\psi] , \quad (14)$$

$$\delta p = \frac{1}{a^2}\phi'(\delta\phi' - \phi'\Phi) - V_{,\phi}\delta\phi - \frac{1}{a^2}\psi'(\delta\psi' - \psi'\Phi) - W_{,\psi}\delta\psi , \quad (15)$$

respectively. The linear terms in the equations obtained by varying the matter action with respect to the two matter fields, we obtain the equations of motion for these scalar field perturbations,

$$\delta\phi'' = -2\mathcal{H}\delta\phi' - k^2\delta\phi - a^2V_{,\phi\phi}\delta\phi - 2a^2V_{,\phi}\Phi + 4\phi'\Phi' , \quad (16)$$

$$\delta\psi'' = -2\mathcal{H}\delta\psi' - k^2\delta\psi + a^2W_{,\psi\psi}\delta\psi + 2a^2W_{,\psi}\Phi + 4\psi'\Phi' . \quad (17)$$

3 A Concrete Example of a Quintom Bounce

In order to make the analysis more quantitative, in this section we would like to take a specific example of a Quintom model. Moreover, our Quintom model should not only provide a bouncing solution of the universe, but also realize a scenario where the EoS crosses the cosmological constant boundary and yields a normal (i.e. non-Phantom-like) evolution of the universe at very early and very late times. Consequently, we demand that the ghost field ψ dominates only close to the bounce point. It must be able to decay after the bounce.

3.1 Background

We will assume vanishing potential for the ghost field, whose background energy density thus consists of kinetic energy exclusively. From the equation of motion of the ghost field it follows that $\dot{\psi}$ evolves proportional to a^{-3} . Thus, the kinetic energy density of the ghost field is proportional to a^{-6} . Therefore, the contribution of the ghost field ψ will dominate at the bounce point, but will decay quickly before and afterwards. This is just what we need in order to achieve both a bounce and connect to the observed universe.

We thus take the Lagrangian of Quintom model to be

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - V(\phi) , \quad (18)$$

and, to be specific, we choose a simple form of the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 . \quad (19)$$

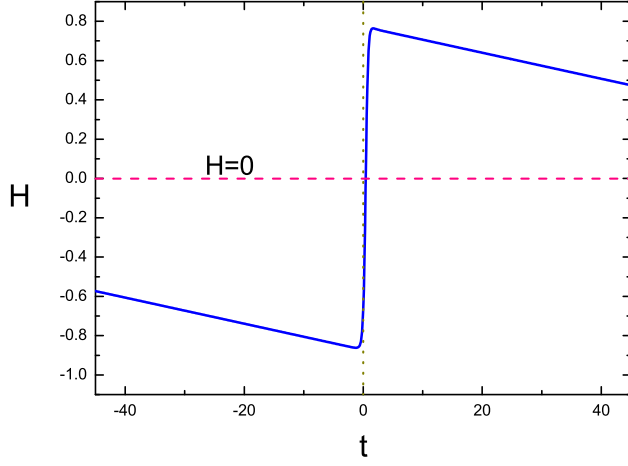


Figure 1: Plot of the evolution of the Hubble parameter H in the model (18). In the numerical calculation we choose the initial values of parameters as: $\phi = -5.6 \times 10^3$, $\dot{\phi} = 2.56 \times 10^2$, $\dot{\psi} = 4.62 \times 10^{-73}$, $m = 1.414 \times 10^{-1}$. The initial time was chosen to be $t = -500$. Note that in this and the following figures with results from numerical simulations, all masses are expressed in units of $10^{-6}M_{pl}$.

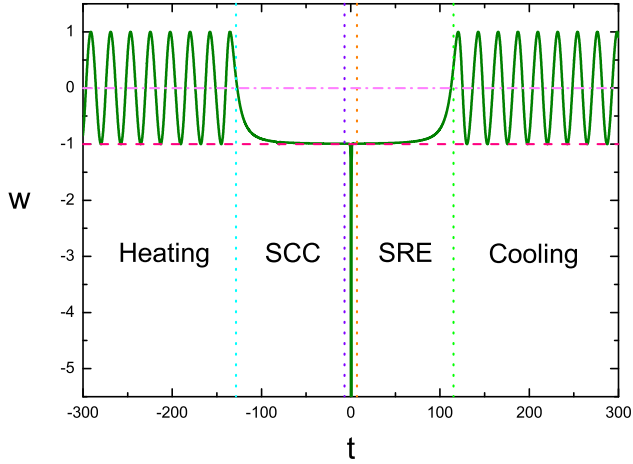


Figure 2: Plot of the evolution of the EoS parameter w in the model (18). The initial values of parameters are the same as in Fig. 1.

In this model, the evolution of the universe is symmetric about the bounce point. In the region long before the bounce point, we choose initial conditions for which the energy density in the field ϕ dominates by a large factor. Thus, in the initial phase of evolution (Period 1) the field ϕ and hence also the EoS of the universe will oscillate about $w = 0$ and the average state is similar to that in a matter dominated period. Since the universe is contracting, it is heating up. Once the amplitude of the field oscillation gets sufficiently large and the absolute value of the Hubble expansion parameter increases sufficiently, the universe will enter a nearly de-Sitter contracting phase (Period 2) with $w \simeq -1$. This transition is the time reversal of the end of the slow-rolling period in scalar field inflation. During this second period, the period of quasi-de-Sitter contraction, the field ϕ climbs slowly up its potential. During both Periods 1 and 2, but in particular in Period 2, the contribution of ψ to the energy density is increasing. We take initial conditions such that at the transition point between Periods 1 and 2 the contribution of ψ is still subdominant. Due to the exponential decrease of the scale factor during Period 2, the contribution of ψ rises exponentially and soon starts to dominate. At this point, the EoS parameter w crosses the barrier $w = -1$. This triggers the onset of Period 3, the bouncing phase. The parameter w approaches negative infinity at which time the bounce occurs. After the bounce, the time reverse of the evolution before the bounce occurs. The universe again enters a quasi-de-Sitter phase (Period 4), but this time the universe is expanding nearly exponentially, i.e., a period of inflation takes place. During this period, ϕ is rolling slowly but the ghost field decays rapidly, and the EoS returns to a non-Phantom-like value $w > -1$. Once ϕ decreases below a critical value (the same critical value which signifies the end of inflation in an $m^2\phi^2$ inflation model), it will start to oscillate about the minimum of its potential, and the EoS starts to once again oscillate around $w = 0$.

In Figs. 1 and 2 we plot the evolution of the Hubble parameter and of the EoS, calculated numerically. In this calculation, we normalized the dimensional parameters such as m , H , ϕ and ψ by dividing by a mass scale M which we took to be $10^{-6}M_{pl}$ where the Planck mass is determined in terms of Newton's gravitational constant by $M_{pl} = 1/\sqrt{G}$. From Fig. 2, we can see that in this model the universe undergoes a heating phase, a slow-climb-contracting (SCC) phase, a bounce, a slow-roll-expanding (SRE) phase, and finally a phase of cooling. The transitions between the phases are smooth.

Note that the background evolution obtained above is rather generic and does not depend sensitively on the details of the initial conditions. The reason is the following: In our model we have introduced a Phantom field which makes it rather trivial to obtain a bouncing solution. Since we have chosen the potential of the Quintom field to vanish, i.e. $W(\psi) = 0$, the Phantom field evolves as $\dot{\psi} \propto a^{-3}$ which is similar to matter with its EoS parameter equal to 1. Thus, we can conclude that a bouncing solution is obtained if the absolute value of the Phantom energy density in the contracting universe grows faster than that of the normal matter. This will occur if the EoS of the normal matter is smaller than 1. In our concrete model we have chosen normal matter to be a canonical field ϕ with a quadratic potential $V(\phi) = m^2\phi^2/2$, which leads to an EoS parameter smaller than 1. If the initial conditions are imposed during the period when the Phantom part is negligible, the parameters of the canonical field such as the mass m and its initial conditions only affect how long the universe is staying in the heating phase when w is oscillating around 0 and in the SCC phase when ($w \simeq -1$).

3.2 Perturbations

To study the evolution of the perturbation, we combine the Eqs. (9), (10) and (11) and obtain the equation of motion of the gravitational potential

$$\Phi'' + 2(\mathcal{H} - \frac{\phi''}{\phi'})\Phi' + 2(\mathcal{H}' - \mathcal{H}\frac{\phi''}{\phi'})\Phi - \nabla^2\Phi = 8\pi G(2\mathcal{H} + \frac{\phi''}{\phi'})\psi'\delta\psi. \quad (20)$$

Since there is only a single physical scalar field degree of freedom for adiabatic scalar metric fluctuations, all other perturbation variables can be determined from Φ , knowing the evolution of the background. A frequently used variable is ζ , the curvature fluctuation in comoving coordinates, which is given by

$$\zeta \equiv \Phi + \frac{\mathcal{H}}{\mathcal{H}^2 - \mathcal{H}'}(\Phi' + \mathcal{H}\Phi). \quad (21)$$

In the case of an expanding universe with regular matter, this variable is known to well describe the adiabatic fluctuations on large scales since it is conserved on super-Hubble scales because it obeys the equation

$$(1+w)\zeta' = \frac{2k^2(\Phi' + \mathcal{H}\Phi)}{9\mathcal{H}^2}.$$

However, this equation becomes singular both when the universe is at the bounce point when the Hubble parameter H approaches 0 and also when the universe is crossing the phantom boundary $w = -1$. Therefore, ζ is ill-defined near the bounce point, as already remarked in [40, 41]. Thus, we will focus on the evolution of Φ which is well defined throughout the bounce. We are focusing on adiabatic fluctuations only in this paper. Since there are two matter fields, it is possible to have entropy fluctuations. We hope to return to this question in future work.

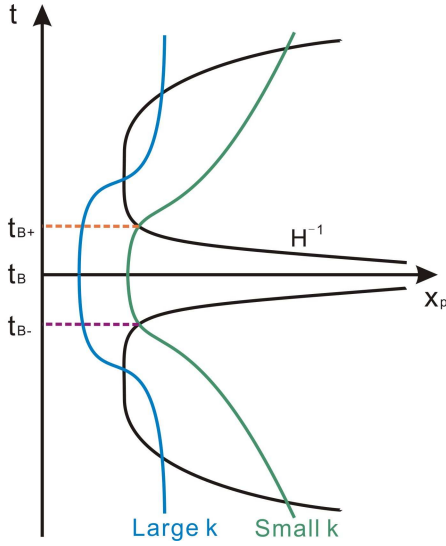


Figure 3: A sketch of the evolution of perturbations with different comoving wave numbers k in the Quintom Bounce.

Since in the linear theory of cosmological perturbations all Fourier modes evolve independently, we will follow each mode, labelled by its comoving wave number k , independently. Interestingly, in our model there are four times when a perturbation mode crosses the Hubble radius. The evolution of scales

is sketched in Fig. 3. Perturbations start out inside the Hubble radius in the far past. Since in the initial heating phase the Hubble radius is shrinking faster than the scale factor, modes exit the Hubble radius. Once Period 2, the "slow climb phase" starts, the scale factor shrinks nearly exponentially whereas the Hubble radius is constant. Thus, wavelengths which are not too large will re-enter the Hubble radius during this phase. During the rest of this phase, and throughout the bounce period, the scale remains sub-Hubble, and exits the Hubble radius again in the post-bounce inflationary phase, only to re-enter the Hubble radius at late times. Modes with an extremely long wavelength will remain outside the Hubble radius throughout Period 2. Since the Hubble radius diverges at the bounce point, even these scales will enter the Hubble radius before the bounce.

To summarize, scales k fall into two broad categories, as sketched in Fig. 3. The first consists of modes with comoving wave number k large enough for the perturbations to return inside the Hubble radius before the bouncing phase and escape outside after the bouncing phase; the second consists of modes with small k such that they enter the Hubble radius during the bouncing phase. We will show later that there is a difference in the post-bounce spectra of fluctuations between these two range of scales these.

In a bouncing universe, it is unsuitable to use the equation of motion for Mukhanov-Sasaki variable v [42, 43] (related closely to ζ) because this variable is singular both at the bounce point and the cosmological constant boundary [40, 41, 18]. Therefore, in the following we will directly study the behavior of perturbations from Eq. (20).

In the following we will develop analytical approximations to study the evolution of the fluctuations in the various phases. We will denote the time corresponding to the end of inflation by t_c , the end of the bounce phase by t_B .

3.2.1 Heating Phase

Long before the bounce, when $t \ll -t_B$, we have an average value of the EoS parameter which is $w = 0$, and the contribution of ψ can be neglected. In this region, we can treat the evolution of the universe as being dominated by a single field ϕ , with the Friedmann equations being

$$\mathcal{H}^2 \simeq \frac{8\pi G}{3} \left(\frac{1}{2} \phi'^2 + a^2 V \right), \quad (22)$$

$$\mathcal{H}' \simeq \frac{8\pi G}{3} (-\phi'^2 + a^2 V), \quad (23)$$

using conformal time and Hubble parameter. Further we have

$$\frac{\phi''}{\phi'} = \frac{2\mathcal{H}\mathcal{H}' - \mathcal{H}''}{2(\mathcal{H}^2 - \mathcal{H}')} , \quad (24)$$

and $a \propto \eta^2$, with $\eta < 0$ during the contracting phase. Thus, we obtain the expressions

$$\mathcal{H} = \frac{2}{\eta}, \quad \mathcal{H}' = -\frac{2}{\eta^2}, \quad \mathcal{H}'' = \frac{4}{\eta^3}. \quad (25)$$

Making use of these relations, we obtain the following equation for the gravitational potential

$$\Phi_k'' + \frac{6}{\eta} \Phi_k' + k^2 \Phi_k = 0, \quad (26)$$

in momentum space. Note that this equation is very similar to the perturbation equation in a matter dominant period except that the last term $k^2 \Phi_k$ does not exist for hydrodynamical perturbations (ordinary

hydrodynamical matter is non-relativistic, our scalar field matter is relativistic). The analytical solution of the above equation of motion can be written as follows,

$$\Phi_k = \eta^{-\frac{5}{2}} [k^{-\frac{5}{2}} A_k J_{\frac{5}{2}}(k\eta) + k^{\frac{5}{2}} B_k J_{-\frac{5}{2}}(k\eta)] , \quad (27)$$

where the coefficients A_k and B_k can be determined from the initial condition.

To set the initial conditions, we first define the variable

$$u_k \equiv \frac{a\Phi_k}{4\pi G\phi'} . \quad (28)$$

This is the variable in terms of which the initial conditions are determined by canonical quantization [38]. We will take the Bunch-Davis vacuum as the initial conditions (when η is sufficiently far away from the bounce point)^{††}:

$$u_k = \frac{1}{\sqrt{2k^3}} e^{-ik\eta} . \quad (29)$$

During Period 1 when $w = 0$ we have $\frac{\phi'}{a} \sim \eta^{-3}$. Therefore the initial condition for the gravitational potential becomes

$$\Phi_k = A\eta^{-3} \frac{e^{-ik\eta}}{\sqrt{2k^3}} , \quad A \equiv 4\pi G\sqrt{\rho_0}\eta_0^3 , \quad (30)$$

where A is a normalization factor. When $|k\eta| \gg 1$ we can take the asymptotical form of the Bessel function and, matching with the Bunch-Davies vacuum, determine the coefficients. The result is

$$A_k = i\frac{\sqrt{\pi}}{2} A k^{\frac{3}{2}} , \quad B_k = -\frac{\sqrt{\pi}}{2} A k^{-\frac{7}{2}} . \quad (31)$$

When the wavelength of perturbation is larger than Hubble radius with $|k| \ll \mathcal{H}$, then by using another asymptotic form of the Bessel functions we obtain the following form of Φ_k :

$$\Phi_k \simeq \frac{iAk^{3/2}}{15\sqrt{2}} - \frac{3A}{\sqrt{2}} k^{-7/2} \eta^{-5} . \quad (32)$$

From this result, we can see that of the two modes of Φ one is constant and the other growing during the contracting heating phase.

3.2.2 Slow-Climb-Contracting (SCC) Phase

For the universe in the contracting phase, we have $H < 0$. In this case, $3H\dot{\phi}$ is anti-frictional, and instead of damping the motion of ϕ in the expanding phase it accelerates the motion of ϕ . Consequently, the canonical scalar field ϕ will eventually stop oscillating and start to climb up its potential. We denote this transition time as η_c . For $(\eta > \eta_c)$ on, the universe is in a period of SCC phase with $w \simeq -1$. We assume that in the beginning of this period the ghost field can still be neglected. Therefore, the equation of motion for the gravitational potential becomes

$$\Phi_k'' + 2\sigma\mathcal{H}\Phi_k' + 2(\sigma - \epsilon)\mathcal{H}^2\Phi_k + k^2\Phi_k = 0 , \quad (33)$$

where we take the slow-climb(slow-roll) parameters as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \quad \text{and} \quad \sigma \equiv -\frac{\ddot{H}}{2H\dot{H}} . \quad (34)$$

^{††}We will discuss other possible initial conditions in a followup paper.

As is well known, we have

$$\eta - \eta_c \simeq 1/\mathcal{H}_c - 1/\mathcal{H} , \quad (35)$$

where a subscript ‘c’ indicates that the corresponding quantity is evaluated at the time η_c .

Making use of the above relations, we can solve the equation (33) for the gravitational perturbation:

$$\Phi_k = (\eta - \tilde{\eta}_c)^\alpha [k^{-\nu} C_k J_\nu(k(\eta - \tilde{\eta}_c)) + k^\nu D_k J_{-\nu}(k(\eta - \tilde{\eta}_c))] , \quad (36)$$

where $\alpha \simeq \frac{1}{2}$, $\nu \simeq \frac{1}{2}$ and $\tilde{\eta}_c \equiv \eta_c + 1/\mathcal{H}_c$. We can obtain the asymptotic form of Eq. (36) in the sub-Hubble region

$$\Phi_k \simeq \sqrt{\frac{2}{\pi}} \left[\frac{C_k}{k} \sin(k(\eta - \tilde{\eta}_c)) + D_k \cos(k(\eta - \tilde{\eta}_c)) \right] , \quad (37)$$

and in the super-Hubble region

$$\Phi_k \simeq \sqrt{\frac{2}{\pi}} [C_k(\eta - \tilde{\eta}_c) + D_k] . \quad (38)$$

In the latter form the two modes are a constant mode D_k and a mode with coefficient C_k which is growing.

Through matching Φ and ζ (these are the matching conditions derived in [36, 35]) in the heating phase and in the SCC phase at the time η_c , we can determine the coefficients C_k and D_k as follows,

$$\begin{aligned} C_k &= -\mathcal{H}_c \left[\frac{1}{15} \left(1 - \frac{2}{3} \epsilon \right) A_k + 3(1 + \epsilon) B_k \eta_c^{-5} \right] , \\ D_k &= \epsilon \left(\frac{2}{45} A_k - 3 B_k \eta_c^{-5} \right) \simeq 0 . \end{aligned} \quad (39)$$

Therefore, in the SCC phase the main contribution to the perturbation comes from the initial growing mode of Φ_k .

3.2.3 Bouncing Phase

Once the universe enters the quasi-de-Sitter contracting phase, the contribution of the ghost field ψ becomes more and more important, since its kinetic energy density scales as a^{-6} . Therefore, the SCC phase will cease at some moment η_{B-} when the contribution of ψ begins to dominate, and then the EoS parameter of the universe will cross $w = -1$ and fall to negative infinity rapidly. Correspondingly, the energy density and the Hubble parameter reach zero when the bounce takes place at the time η_B . Since at this moment the Hubble radius $1/H$ approaches infinity, all wavelengths will be inside the Hubble radius. Thus, the perturbations will be once again oscillating close to the bounce point. Short wavelength modes will have time to perform many oscillations which are sub-Hubble near the bounce point, whereas long wavelength modes (which have a longer intrinsic period and also spend less time being sub-Hubble) will not have time to perform a complete oscillation. As we will see, this difference leads to a difference in the way the initial spectrum of fluctuations transfers through the bounce.

It is rather complicated to solve the perturbation equation exactly from Eq. (20). In order to gain some analytical insight into the evolution of fluctuations in this region, we resort to some approximations in order to simplify the equation.

A convenient parametrization of the Hubble parameter near the bounce point (we choose $t = 0$ to correspond to the bounce point) is

$$H(t) = \alpha t , \quad (40)$$

where α is a positive constant.

In this case, we obtain the analytic forms of the scale factor and the comoving Hubble parameter:

$$a = \frac{a_B}{1 - \frac{2}{\pi}\alpha a_B^2(\eta - \eta_B)^2}, \quad \mathcal{H} = \frac{\frac{4}{\pi}\alpha a_B^2(\eta - \eta_B)}{1 - \frac{2}{\pi}\alpha a_B^2(\eta - \eta_B)^2}, \quad (41)$$

where a_B denotes the scale factor at the bounce point.

In the bouncing process, the field ϕ is initially climbing up to the hill. However, as the absolute value of H is decreasing, $\ddot{\phi}$ changes its sign which then leads to the field ϕ reversing its direction and beginning to slowly roll down the potential. At this stage, the universe has bounced and H is positive.

Making use of the parameterization $H = \alpha t$ (see (40)) we can solve Eq. (4) to obtain

$$\frac{\ddot{\phi}}{\dot{\phi}} = -3\alpha t \dot{\psi}^2 / (\dot{\psi}^2 - \frac{\alpha}{4\pi G}) \simeq -3H, \quad (42)$$

around the bounce point. Consequently, we can derive another formula $\phi'' \simeq -2\mathcal{H}\phi'$. In Sec. 3.2.5, we will show that this is a very successful simulation by comparing with the numerical calculation. In order to show this point more explicitly, in Fig. 4 we present a plot of the factor $3H + \frac{\ddot{\phi}}{\dot{\phi}}$ in the bouncing phase. The values shown in the figure support the approximation we have made in Eq. (42) since from the figure, it can be seen that this factor vanishes around the bounce point.

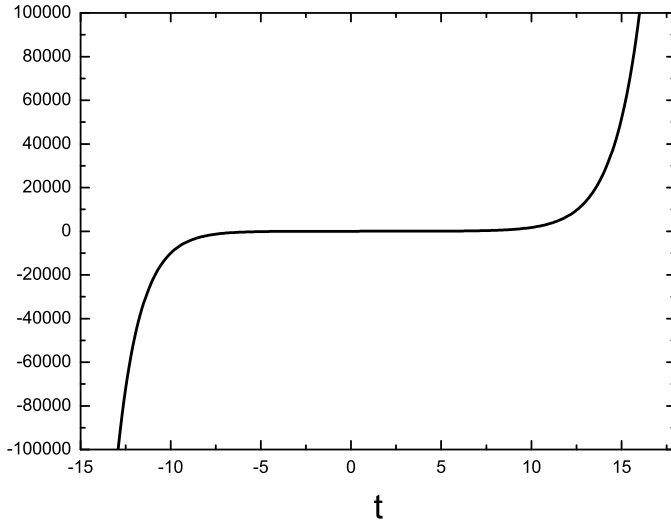


Figure 4: A plot of the evolution of the factor $3H + \frac{\ddot{\phi}}{\dot{\phi}}$ in the bouncing phase.

We suppose that the approximations above should be valid only in the neighborhood of the bounce point, and hence the square term and higher order terms of $|\eta - \eta_B|$ should be neglected.

Substituting these approximations into Eq. (20), we obtain the following equation:

$$\Phi_k'' + 3y_1(\eta - \eta_B)\Phi_k' + (k^2 + y_1)\Phi_k \simeq 0, \quad (43)$$

where we define the parameter $y_1 \equiv \frac{8}{\pi}\alpha a_B^2$. The solution to this equation can be written as

$$\Phi_k = e^{-\frac{3}{2}y_1(\eta - \eta_B)^2} \left[E_k H_{\bar{m}}(\sqrt{3y_1/2}(\eta - \eta_B)) + F_k {}_1F_1\left(-\frac{\bar{m}}{2}; \frac{1}{2}; \frac{3}{2}y_1(\eta - \eta_B)^2\right) \right], \quad (44)$$

which is constructed by a \bar{m} -th Hermite polynomial and a confluent hyper-geometric function with $\bar{m} = \frac{k^2 - 2y_1}{3y_1}$ and two undetermined coefficients E_k, F_k . These two functions are linearly independent, but both of them show oscillating behavior when y_1/k^2 is sufficiently small. This is to be expected in our bounce phase since the Hubble radius blows up at the bounce point and all scales become sub-Hubble and hence the associated mode functions should oscillate, even for $|k(\eta - \eta_B)| \ll 1$.

If the parameter \bar{m} is big enough, We can make use of the following approximate

$$\Phi_k \simeq e^{-\frac{3}{4}y_1(\eta - \eta_B)^2} \left\{ \tilde{F}_k \cos[k(\eta - \eta_B)] + \tilde{E}_k \sin[k(\eta - \eta_B)] \right\}, \quad (45)$$

where

$$\tilde{E}_k \equiv -\frac{2^{\bar{m}+1}\sqrt{\pi}}{\sqrt{2\bar{m}}\Gamma(-\frac{\bar{m}}{2})}E_k \quad (46)$$

and

$$\tilde{F}_k \equiv \frac{2^{\bar{m}}\sqrt{\pi}}{\Gamma(\frac{1-\bar{m}}{2})}E_k + F_k. \quad (47)$$

Interestingly, we obtain a factor $e^{-\frac{3}{4}y_1(\eta - \eta_B)^2}$ which can enlarge the amplitude of the perturbation before the bounce and suppress it after the bounce. We sketch this solution in Fig. 5 and will find that it is consistent with the numerical results in Sec. 3.2.5.

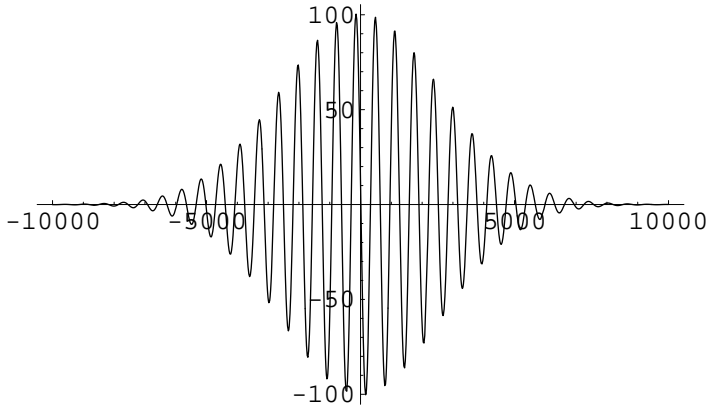


Figure 5: A sketch of the evolution of the gravitational potential Φ in the bouncing phase.

Note that there are two possible ways for the modes in the SCC phase to evolve into the bouncing phase, as described in the beginning of Sec. 3.2 (also see Fig. 3). The first possibility is that the comoving wave number k is large enough such that the mode reenters the Hubble radius in the SCC phase. The second is that k is small enough and the mode re-enters the Hubble radius only during the bouncing phase.

In the first case with $k \gg \mathcal{H}_{B-}$, the spectrum has been oscillating already at late times in the SCC phase. Through matching Φ and ζ in the SCC phase and in the bouncing phase at the time η_{B-} we can determine the coefficients $\tilde{E}_k^{(1)}$ and $\tilde{F}_k^{(1)}$ as follows,

$$\begin{aligned} \tilde{E}_k^{(1)} \simeq & e^{\frac{3}{4}y_1(\eta_{B-} - \eta_B)^2} \sqrt{\frac{2}{\pi}} \left\{ \left(\frac{C_k}{k} \sin[k(\eta_{B-} - \tilde{\eta}_c)] + D_k \cos[k(\eta_{B-} - \tilde{\eta}_c)] \right) \sin[k(\eta_{B-} - \eta_B)] \right. \\ & \left. - \frac{1}{k(\eta_{B-} - \eta_B)} \frac{1}{\epsilon \mathcal{H}_{B-}} [(C_k + \mathcal{H}_{B-} D_k) \cos[k(\eta_{B-} - \tilde{\eta}_c)] \right\} \end{aligned}$$

$$+\left(\frac{\mathcal{H}_{B-}}{k}C_k - kD_k\right)\sin[k(\eta_{B-} - \tilde{\eta}_c)]\cos[k(\eta_{B-} - \eta_B)]\Big\} , \quad (48)$$

$$\begin{aligned} \tilde{F}_k^{(1)} &\simeq e^{\frac{3}{4}y_1(\eta_{B-}-\eta_B)^2}\sqrt{\frac{2}{\pi}}\Big\{\left(\frac{C_k}{k}\sin[k(\eta_{B-} - \tilde{\eta}_c)] + D_k\cos[k(\eta_{B-} - \tilde{\eta}_c)]\right)\cos[k(\eta_{B-} - \eta_B)] \\ &+ \frac{1}{k(\eta_{B-} - \eta_B)}\frac{1}{\epsilon\mathcal{H}_{B-}}[(C_k + \mathcal{H}_{B-}D_k)\cos[k(\eta_{B-} - \tilde{\eta}_c)] \\ &+ \left(\frac{\mathcal{H}_{B-}}{k}C_k - kD_k\right)\sin[k(\eta_{B-} - \tilde{\eta}_c)]\sin[k(\eta_{B-} - \eta_B)]\Big\} . \end{aligned} \quad (49)$$

From this result we can read off that, in the case when $k \gg \mathcal{H}_{B-}$ the coefficients of the perturbations in bouncing phase depend in an UN-suppressed way on the coefficient C_k of the growing mode in the SCC phase. However, this is not an absolute relation since the coefficient D_k of the constant mode in the SCC phase may play an important role when we fine-tune the phases $k(\eta_{B-} - \eta_c)$ and $k(\eta_{B-} - \eta_B)$.

Secondly, we consider the case when $k \sim \mathcal{H}_{B-}$. Through matching the spectrum in the bouncing phase with that in the SCC phase, the coefficients can be determined as follows,

$$\begin{aligned} \tilde{E}_k^{(2)} &\simeq e^{\frac{3}{4}y_1(\eta_{B-}-\eta_B)^2}\sqrt{\frac{2}{\pi}}\Big\{\left(D_k - \frac{C_k}{\mathcal{H}_{B-}}\right)\sin[k(\eta_{B-} - \eta_B)] \\ &- \frac{1}{k(\eta_{B-} - \eta_B)}\frac{D_k}{\epsilon}\cos[k(\eta_{B-} - \eta_B)]\Big\} \\ &\simeq -e^{\frac{3}{4}y_1(\eta_{B-}-\eta_B)^2}\frac{\sqrt{2/\pi}}{k(\eta_{B-} - \eta_B)}\frac{D_k}{\epsilon}\cos[k(\eta_{B-} - \eta_B)] , \end{aligned} \quad (50)$$

$$\begin{aligned} \tilde{F}_k^{(2)} &\simeq e^{\frac{3}{4}y_1(\eta_{B-}-\eta_B)^2}\sqrt{\frac{2}{\pi}}\Big\{\left(D_k - \frac{C_k}{\mathcal{H}_{B-}}\right)\cos[k(\eta_{B-} - \eta_B)] \\ &+ \frac{1}{k(\eta_{B-} - \eta_B)}\frac{D_k}{\epsilon}\sin[k(\eta_{B-} - \eta_B)]\Big\} \\ &\simeq e^{\frac{3}{4}y_1(\eta_{B-}-\eta_B)^2}\frac{\sqrt{2/\pi}}{k(\eta_{B-} - \eta_B)}\frac{D_k}{\epsilon}\sin[k(\eta_{B-} - \eta_B)] . \end{aligned} \quad (51)$$

From these two equations we find that it is usually the coefficient D_k of the sub-dominant mode in the SCC phase which mainly dominates the coefficients of the perturbations in the bouncing phase, unless we fine-tune the phase $k(\eta_{B-} - \eta_B)$.

3.2.4 Slow-Roll-Expanding Phase (SRE)

Due to symmetry of the Quintom model, the scalar field ϕ begins to slow roll down to the bottom of its potential and the ghost field ψ decays soon after the bounce. Therefore, a phase of slow-roll expansion (like slow-roll inflation) starts at the time η_{B+} . We denote the scale factor at this moment as a_{B+} . During this period, the Hubble parameter is nearly constant which is similar to what occurs in the usual inflationary scenario. Interestingly, the EoS of the universe evolves from below -1 to above which provides a link with the normal expanding history of our universe.

The equation of motion for the gravitational potential in this phase is the same as in the SCC phase (Eq. (33)). Thus, the solutions takes the form

$$\Phi_k = (\eta - \tilde{\eta}_{B+})^\alpha [k^{-\nu} G_k J_\nu(k(\eta - \tilde{\eta}_{B+})) + k^\nu H_k J_{-\nu}(k(\eta - \tilde{\eta}_{B+}))] , \quad (52)$$

where $\tilde{\eta}_{B+} \equiv \eta_{B+} + 1/\mathcal{H}_{B+}$ and the values of α and ν are the same as those which appeared in Sec.

3.2.2. We can obtain the asymptotical form of Eq. (52) in the sub-Hubble region

$$\Phi_k \simeq \sqrt{\frac{2}{\pi}} \left[\frac{G_k}{k} \sin(k(\eta - \tilde{\eta}_{B+})) + H_k \cos(k(\eta - \tilde{\eta}_{B+})) \right] \quad (53)$$

and in the super-Hubble region

$$\Phi_k \simeq \sqrt{\frac{2}{\pi}} [G_k(\eta - \tilde{\eta}_{B+}) + H_k]. \quad (54)$$

The second mode is constant (it corresponds to the usual dominant mode on super-Hubble scales in an expanding universe) and the first mode is decreasing.

When determining the coefficients of the two perturbation modes, we again need to match with the perturbation in bouncing phase in two cases. In the case $k \gg \mathcal{H}_{B+}$ and assuming that all the phases are nontrivial, we obtain

$$G_k^{(1)} \simeq e^{-\frac{3}{4}y_1(\eta_{B+}-\eta_B)^2} \frac{\sqrt{\pi/2}}{2 \sin[k(\eta_{B+}-\tilde{\eta}_{B+})]} k \left\{ \tilde{E}_k \sin[k(\eta_{B+}-\eta_B)] + \tilde{F}_k \cos[k(\eta_{B+}-\eta_B)] \right\}, \quad (55)$$

$$H_k^{(1)} \simeq e^{-\frac{3}{4}y_1(\eta_{B+}-\eta_B)^2} \frac{\sqrt{\pi/2}}{2 \cos[k(\eta_{B+}-\tilde{\eta}_{B+})]} \left\{ \tilde{E}_k \sin[k(\eta_{B+}-\eta_B)] + \tilde{F}_k \cos[k(\eta_{B+}-\eta_B)] \right\}. \quad (56)$$

Therefore, in this case both coefficients are of the same order which means that the two modes are both important in the sub-Hubble region. However, as soon as the perturbation evolves outside the Hubble radius, the decreasing mode becomes negligible. The amplitude of the dominant mode in the expanding phase is set by the amplitude of the growing mode in the contracting phase. We will compare our result with that obtained in other works on non-singular bouncing cosmologies in the Discussion section.

In the other case we take $k \sim \mathcal{H}_{B+}$ and also assume all the phases are nontrivial. We then obtain

$$G_k^{(2)} \simeq -e^{-\frac{3}{4}y_1(\eta_{B+}-\eta_B)^2} \mathcal{H}_{B+} \sqrt{\frac{\pi}{2}} \left\{ \tilde{E}_k \sin[k(\eta_{B+}-\eta_B)] + \tilde{F}_k \cos[k(\eta_{B+}-\eta_B)] \right\}, \quad (57)$$

$$\begin{aligned} H_k^{(2)} &\simeq -e^{-\frac{3}{4}y_1(\eta_{B+}-\eta_B)^2} (\eta_{B+}-\eta_B) \epsilon \sqrt{\frac{\pi}{2}} k \left\{ \tilde{E}_k \cos[k(\eta_{B+}-\eta_B)] - \tilde{F}_k \sin[k(\eta_{B+}-\eta_B)] \right\} \\ &\simeq 0. \end{aligned} \quad (58)$$

Thus, we obtain the result that for these very long wavelength mode, the perturbations contain, to leading order in k , only the decreasing mode at late time. This is similar to the behavior of fluctuations in models with a singular bounce, such as the Pre-Big-Bang [7] and Ekpyrotic universe [8] scenarios. In the Pre-Big-Bang scenario, it has been shown [44] that the growing mode of the fluctuations in the contracting phase matches (up to correction factors which are suppressed by powers of k) only to the decaying mode in the expanding phase. The amplitude of the dominant mode in the expanding phase is thus set by the amplitude of the constant (and thus sub-dominant) mode in the contracting phase. The same result also holds [45, 46, 47, 48] in the case of four dimensional toy model descriptions of the Ekpyrotic scenario, although in the case of the Ekpyrotic scenario the physical model is defined in higher dimensions, and an analysis of the evolution of fluctuations in the higher dimensional context [49, 50] shows that there is a non-trivial mixing between the growing mode in the contracting phase and the dominant mode in the expanding phase.

3.2.5 Numerical Results

In the above sections we have present the analytic calculation of our model, during the process we have taken an assumption that the right hand side of Eq. (20) is negligible during the whole evolution. This is

a key to solving the perturbation equation in our model explicitly. As is pointed out in previous sections, the energy density of the phantom field is usually enough small in the heating phase, the SCC phase, the SRE phase and the cooling phase and hence we have $\psi' \simeq 0$. Therefore it is a good approximation to neglect the right hand side of Eq. (20) in these four phases. Moreover, we have argued that the relation $2\mathcal{H} + \frac{\phi''}{\phi'} \simeq 0$ in the bouncing phase which makes the right hand side of Eq. (20) negligible. To make sure the assumption valid during the whole evolution, we plot the ratio of the r.h.s to the term $k^2\Phi$ appearing in the equation as shown in Fig. 6. From this figure we can find that the value of the ratio in our model is always much less than 1 which agrees with our assumption very well.

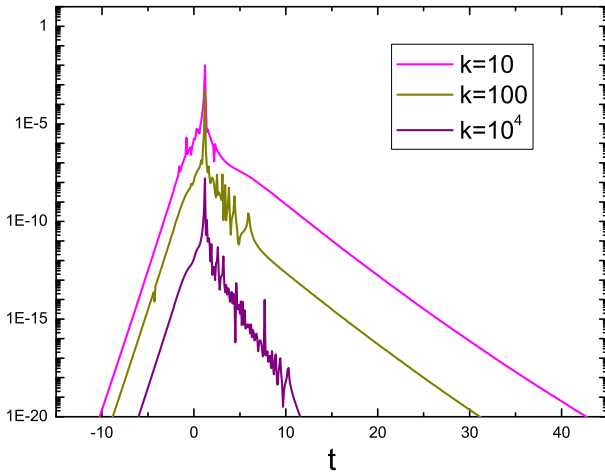


Figure 6: Plot of the evolution of the absolute value of the ratio of the r.h.s to the term $k^2\Phi$ in Eq. (20). The initial values of the background parameters are the same as in Fig. 1.

In order to make the previous analysis exact and precise, we also solved the perturbation equations numerically. We provide the numerical result in Fig. 7, Fig. 8, and Fig. 9. In Fig. 7, we plot the absolute value of the Newtonian gravitational potential $|\Phi|$ for different values of k . We can see from this figure that in the period far before the bounce point, those curves are similar. At the beginning of the evolution, when the wavelength is sub-Hubble, the perturbation is near its Bunch-Davies vacuum and its value oscillates with increasing amplitude. One may notice that the initial oscillatory behaviors of different modes are the same as others which only depend on the background evolution in Fig. 7. We give the explanations as follows. In our model, the canonical field ϕ initially oscillates around its vacuum in the contracting period which results in a heating phase. During this phase, the EoS oscillates between -1 and 1 as shown in Fig. 2. In our analytical calculation we simply take the average value of the EoS $w = 0$ and show that there are two modes for the gravitational potential with one being constant and the other being growing mode. However, when we take the numerical calculation, we encounter the case that there has to be a point satisfying $\dot{\phi} = 0$ and $w = -1$ in every oscillation. At this moment, there can indeed be a large growth of fluctuations due to parametric resonance. Therefore it seems that there are oscillations on the evolution of the gravitational potential, but it is not true. Since these resonances are determined by the background evolution, those “oscillations” have the same frequency and features

regardless of the wave number k . By the same reason we can understand the periodical amplifications of $|\zeta|$ as they appear in Fig. 9. This issue will be discussed in future work. After the mode exits the Hubble radius, the perturbation will be dominated by its growing mode, as is shown in Eq. (32). When it enters into the SCC phase, it grows monotonically at first and becomes oscillatory near the bounce point after the wavelength re-enters the Hubble radius.

During the bouncing phase, the curves of different k -modes behave very differently. For large k , the oscillatory behavior starts earlier and lasts longer because the wavelength of such modes enters into the sub-Hubble region much earlier before the bounce and escapes from the Hubble radius much later after the bounce. The amplitude of the oscillations near the bounce point increases before the bounce and decreases after the bounce, which follows from the Hubble term in the equation of motion which acts as anti-friction in the contracting phase and friction in the expanding phase, as shown in Eq. (45). On the other hand, the oscillating period for small k -modes becomes much shorter since these modes do not enter in the sub-Hubble region until after the bouncing phase has started.

Finally, after the bounce the universe enters into the SRE phase and the wavelengths of the fluctuations become super-Hubble once again. Now, there are also two very different evolution scenarios for fluctuations with different k -modes. In the expanding phase, for large enough values of k , the gravitational potential is dominated by the constant mode; while in the case of small values of k , it is dominated by decreasing mode.

We also plot the values of Φ of different modes around the bounce point in Fig. 8. We can see that, for larger values of k , the oscillations begins much earlier with their amplitude being enlarged at the bounce point, whereas for smaller values of k there is only a very short period of oscillation (with amplified amplitude) which is caused by bounce.

After the bounce, the gravitational potential for large k evolves almost as a constant. However, for small k there is a decrease on the gravitational potential. All these features are consistent with the analytical results discussed above.

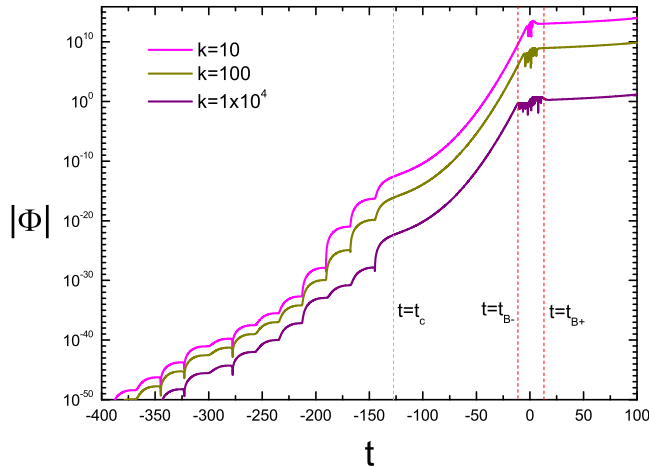


Figure 7: Plot of the evolution of the absolute value of the gravitational potential $|\Phi|$ in our model. The initial values of the background parameters are the same as in Fig. 1.

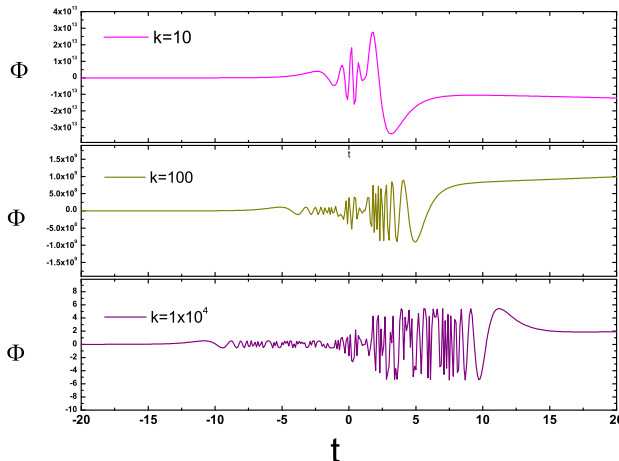


Figure 8: Plot of the evolution of the gravitational potential Φ in our model. The initial values of the background parameters are the same as in Fig. 1.

Finally, we also provide a plot of the curvature perturbation ζ in Fig. 9. From this figure we can see that there inevitably are oscillations of the perturbations near the bounce point.

4 Discussion And Conclusions

Bouncing non-singular cosmologies, with an initial contracting phase which lasts until to a non-vanishing minimal radius is reached and then smoothly transits into an expanding phase, provide a possible solution to the singularity problem of Standard Big Bang cosmology, a problem which is not cured by scalar-field-driven inflationary models.

In this paper, we have considered the evolution of cosmological fluctuations in a particular bouncing cosmology, namely the Quintom Bounce. In this model, space-time continues to be described by Einstein's General Relativity. The violations of the null energy condition required to get a bounce are obtained by making use of Quintom matter, which allows a transition of the EoS w through the cosmological constant boundary. We have studied the evolution of the gravitational potential both analytically and numerically in a particular model of a Quintom Bounce in which matter consists of two scalar fields, one of them a Quintessence-like field, the other a Phantom-like field. We have found that the evolution of the gravitational potential Φ is completely non-singular. Whereas in models with a singular bounce such as the Pre-Big-Bang [7] and the Ekpyrotic [8] scenarios, the perturbations must be matched across a singular space-like surface, which leads to serious conceptual problems [37], in our case no such problems arise^{‡‡}. In an analytical approximation scheme, we can apply matching conditions at non-singular surfaces which are then well justified, and we can also study the evolution numerically without any approximations.

Our second main result is that for short wavelength modes, modes which re-enter the Hubble radius in the slow contracting phase, the dominant fluctuation mode is determined by the growing mode in the

^{‡‡}Recently, a ghost condensation mechanism to smooth out the singularity in the Ekpyrotic scenario has been introduced [51, 52], and it has been shown in this context, using the techniques developed in [53, 54, 55] that making use of iso-curvature modes, the dominant spectrum in the contracting phase can be passed on to the dominant spectrum in the expanding phase.

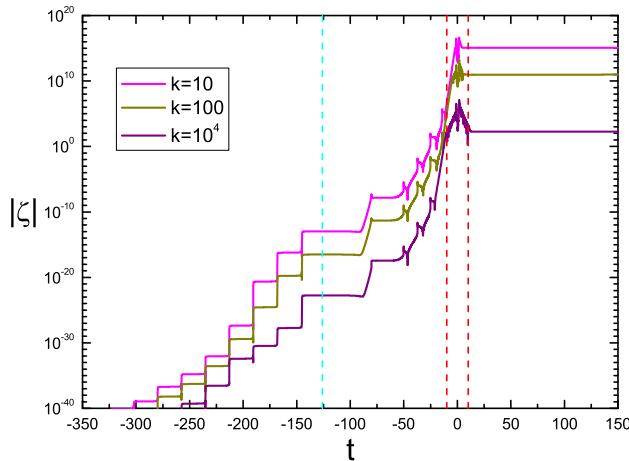


Figure 9: Plot of the evolution of the absolute value of the curvature perturbation $|\zeta|$ in our model. The initial values of the background parameters are the same as in Fig. 1.

contracting phase. This result is similar to what has recently been found in other non-singular bouncing cosmologies [40, 41, 56]. The first of these is a bounce in the context of mirage cosmology, the second in the context of a specific higher derivative gravity model, and the third in the context of K-essence. Earlier work on fluctuations in models with a kinetic term with opposite sign generating a nonsingular bounce is in [57]. The authors of [57] observed that the way the post-bounce spectrum depends on the pre-bounce spectrum depends sensitively on the details of the bounce. In this light, it is not surprising that other bounces show that the dominant mode in the pre-bounce phase does not couple to the dominant mode in the post-bounce phase [11, 12, 58, 59, 60, 61]. For long wavelength modes, however, we find that the dominant fluctuation mode in the contracting phase at leading order in k only sources the decaying mode in the expanding phase, a result similar to what is obtained in singular bouncing models [44, 45, 46, 47, 48].

Thus, the fluctuations in the Quintom Bounce model evolve in a way which combines aspects found in some other recently studied non-singular bouncing cosmologies [40, 41] (the behavior of short wavelength modes) with aspects found in singular bouncing cosmologies (the behavior of the long wavelength modes). Which category of modes determines scales accessible to current cosmological observations depends on the detailed parameters of the model. Our model may help us to understand the physics of singular bounce and non-singular bounce more clearly. As a side remark, let us add that in the Quintom Bounce model there is no trans-Planckian problem [62, 63] for fluctuations since, unless the periods of slow-climb contraction and slow-roll expansion last many Hubble expansion times, the wavelengths of all the observable perturbations will be larger than the Planck length at the bounce point and thus never enter the zone of trans-Planckian ignorance.

Let us close with some more general comments related to the Quintom Bounce. Although this theory is still in the stage of development, there have been works done in the context of a bouncing scenario related to predictions for observations. For example, in [64] (see also [65, 66]) it was pointed out that with a bounce followed by slow-roll inflation it is possible to give a reasonable explanation for the suppression of the low multi-poles of the CMB anisotropies. In [15] and [67] the possibility of obtaining a bounce in

a Quintom model with a single matter field with higher-derivative terms was explored. See also [68, 69] for other works.

Acknowledgments

It is a pleasure to thank Mingzhe Li, Jun-Qing Xia and Yi Wang for helpful discussions. We also would like to thank the anonymous referee for his/her valuable suggestions on our work. This work is supported in part by the National Natural Science Foundation of China under Grants Nos. 90303004, 10533010, 10675136 and 10775180 and by the Chinese Academy of Science under Grant No. KJCX3-SYW-N2. The work of RB is supported in part by an NSERC Discovery Grant, by funds for the Canada Research Chairs Program, and by an FQRNT Team Grant. RB thanks the KITPC for hospitality during the period when a large part of the work on this project was carried out.

References

- [1] A. H. Guth, Phys. Rev. **D23**, 347 (1981).
- [2] K. Sato, Mon. Not. Roy. Astron. Soc. **195**, 467 (1981).
- [3] A. Albrecht and P. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
- [4] A. D. Linde, Phys. Lett. **B108**, 389 (1982).
- [5] A. Borde and A. Vilenkin, Phys. Rev. Lett. **72**, 3305 (1994) [arXiv:gr-qc/9312022].
- [6] A. A. Starobinsky, Phys. Lett. **B91**, 99 (1980).
- [7] G. Veneziano, Phys. Lett. **B265**, 287 (1991); M. Gasperini, and G. Veneziano, Astropart. Phys. **1**, 317 (1993) [arXiv:hep-th/9211021]; M. Gasperini and G. Veneziano, Phys. Rept. **373**, 1 (2003) [arXiv:hep-th/0207130].
- [8] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, Phys. Rev. **D64**, 123522 (2001) [arXiv:hep-th/0103239]; J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt, and N. Turok, Phys. Rev. **D65**, 086007 (2002) [arXiv:hep-th/0108187]; P. J. Steinhardt and N. Turok, Phys. Rev. **D65**, 126003 (2002) [arXiv:hep-th/0111098].
- [9] M. Bojowald, Phys. Rev. Lett. **86**, 5227 (2001) [arXiv:gr-qc/0102069]; M. Bojowald, Class. Quant. Grav. **19**, 2717 (2002) [arXiv:gr-qc/0202077].
- [10] R. Brustein and R. Madden, Phys. Rev. D **57**, 712 (1998) [arXiv:hep-th/9708046].
- [11] C. Cartier, E. J. Copeland and R. Madden, JHEP **0001**, 035 (2000) [arXiv:hep-th/9910169].
- [12] S. Tsujikawa, R. Brandenberger and F. Finelli, Phys. Rev. D **66**, 083513 (2002) [arXiv:hep-th/0207228].
- [13] T. Biswas, A. Mazumdar and W. Siegel, JCAP **0603**, 009 (2006) [arXiv:hep-th/0508194]; T. Biswas, R. Brandenberger, A. Mazumdar and W. Siegel, arXiv:hep-th/0610274.
- [14] M. R. Setare, Phys. Lett. B **602**, 1 (2004) [arXiv:hep-th/0409055].
- [15] Y. F. Cai, T. Qiu, Y. S. Piao, M. Li and X. Zhang, JHEP **0710**, 071 (2007) [arXiv:0704.1090 [gr-qc]].
- [16] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. **B607**, 35 (2005) [arXiv:astro-ph/0404224].
- [17] G. B. Zhao, J. Q. Xia, H. Li, C. Tao, J. M. Virey, Z. H. Zhu and X. Zhang, Phys. Lett. **B648**, 8 (2007) [arXiv:astro-ph/0612728].
- [18] J. Q. Xia, Y. F. Cai, T. T. Qiu, G. B. Zhao and X. Zhang, arXiv:astro-ph/0703202.
- [19] A. Vikman, Phys. Rev. **D71**, 023515 (2005) [arXiv:astro-ph/0407107].
- [20] W. Hu, Phys. Rev. **D71**, 047301 (2005) [arXiv:astro-ph/0410680].
- [21] R. R. Caldwell and M. Doran, Phys. Rev. **D72**, 043527 (2005) [arXiv:astro-ph/0501104].

- [22] G. B. Zhao, J. Q. Xia, M. Li, B. Feng and X. Zhang, Phys. Rev. **D72**, 123515 (2005) [arXiv:astro-ph/0507482].
- [23] L. R. Abramo and N. Pinto-Neto, Phys. Rev. D **73**, 063522 (2006) [arXiv:astro-ph/0511562].
- [24] M. Kunz and D. Sapone, Phys. Rev. **D74**, 123503 (2006) [arXiv:astro-ph/0609040].
- [25] Z. K. Guo, Y. S. Piao, X. M. Zhang and Y. Z. Zhang, Phys. Lett. **B608**, 177 (2005) [arXiv:astro-ph/0410654].
- [26] X. F. Zhang, H. Li, Y. S. Piao and X. M. Zhang, Mod. Phys. Lett. A **21**, 231 (2006) [arXiv:astro-ph/0501652]; Z. K. Guo, Y. S. Piao, X. Zhang and Y. Z. Zhang, Phys. Rev. D **74**, 127304 (2006) [arXiv:astro-ph/0608165]; Y. F. Cai, H. Li, Y. S. Piao and X. M. Zhang, Phys. Lett. B **646**, 141 (2007) [arXiv:gr-qc/0609039].
- [27] Y. F. Cai, M. Z. Li, J. X. Lu, Y. S. Piao, T. T. Qiu and X. M. Zhang, Phys. Lett. B **651**, 1 (2007) [arXiv:hep-th/0701016].
- [28] M. Z. Li, B. Feng and X. M. Zhang, JCAP **0512**, 002 (2005) [arXiv:hep-ph/0503268].
- [29] C. Armendariz-Picon, JCAP **0407**, 007 (2004) [arXiv:astro-ph/0405267]; H. Wei and R. G. Cai, Phys. Rev. D **73**, 083002 (2006) [arXiv:astro-ph/0603052].
- [30] R. G. Cai, H. S. Zhang and A. Wang, Commun. Theor. Phys. **44**, 948 (2005) [arXiv:hep-th/0505186]; P. S. Apostolopoulos and N. Tetradis, Phys. Rev. D **74**, 064021 (2006) [arXiv:hep-th/0604014]; H. S. Zhang and Z. H. Zhu, Phys. Rev. D **75**, 023510 (2007) [arXiv:astro-ph/0611834]; A. Sheykhi, B. Wang and N. Riazi, Phys. Rev. D **75**, 123513 (2007) [arXiv:0704.0666 [hep-th]]; L. P. Chimento, R. Lazkoz, R. Maartens and I. Quiros, JCAP **0609**, 004 (2006) [arXiv:astro-ph/0605450].
- [31] I. Y. Aref'eva, A. S. Koshelev and S. Y. Vernov, Phys. Rev. D **72**, 064017 (2005) [arXiv:astro-ph/0507067]; S. Y. Vernov, arXiv:astro-ph/0612487; A. S. Koshelev, JHEP **0704**, 029 (2007) [arXiv:hep-th/0701103].
- [32] B. Feng, M. Li, Y. S. Piao and X. Zhang, Phys. Lett. B **634**, 101 (2006) [arXiv:astro-ph/0407432]; H. Li, B. Feng, J. Q. Xia and X. Zhang, Phys. Rev. D **73**, 103503 (2006) [arXiv:astro-ph/0509272]; X. F. Zhang and T. Qiu, Phys. Lett. B **642**, 187 (2006) [arXiv:astro-ph/0603824]; T. Qiu, Y. F. Cai and X. M. Zhang, arXiv:0710.0115 [gr-qc]; X. Zhang and F. Q. Wu, Phys. Rev. D **72**, 043524 (2005) [arXiv:astro-ph/0506310]; H. Wei and R. G. Cai, Phys. Rev. D **72**, 123507 (2005) [arXiv:astro-ph/0509328]; X. Zhang, Phys. Rev. D **74**, 103505 (2006) [arXiv:astro-ph/0609699].
- [33] V. K. Onemli and R. P. Woodard, Phys. Rev. D **70**, 107301 (2004) [arXiv:gr-qc/0406098]; E. O. Kahya and V. K. Onemli, Phys. Rev. D **76**, 043512 (2007) [arXiv:gr-qc/0612026].
- [34] R. Lazkoz and G. Leon, Phys. Lett. B **638**, 303 (2006) [arXiv:astro-ph/0602590]; W. Zhao, and Y. Zhang, Phys. Rev. D **73**, 123509 (2006) [arXiv:astro-ph/0604460]; H. Mohseni Sadjadi and M. Al-imohammadi, Phys. Rev. D **74**, 043506 (2006) [arXiv:gr-qc/0605143]; M. R. Setare, Phys. Lett. B **641**, 130 (2006) [arXiv:hep-th/0611165]; R. Lazkoz, G. Leon and I. Quiros, Phys. Lett. B **649**, 103 (2007) [arXiv:astro-ph/0701353]; M. Libanov, V. Rubakov, E. Papantonopoulos, M. Sami and S. Tsujikawa, arXiv:0704.1848 [hep-th]; M. R. Setare and E. C. Vagenas, arXiv:0704.2070 [hep-th]; H. Zhang and Z. H. Zhu, arXiv:0704.3121 [astro-ph].

- [35] N. Deruelle and V. F. Mukhanov, Phys. Rev. D **52**, 5549 (1995) [arXiv:gr-qc/9503050].
- [36] J. C. Hwang and E. T. Vishniac, Astrophys. J. **382**, 363 (1991).
- [37] R. Durrer and F. Vernizzi, Phys. Rev. D **66**, 083503 (2002) [arXiv:hep-ph/0203275].
- [38] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992).
- [39] R. H. Brandenberger, Lect. Notes Phys. **646**, 127 (2004) [arXiv:hep-th/0306071].
- [40] R. Brandenberger, H. Firouzjahi and O. Saremi, arXiv:0707.4181 [hep-th].
- [41] S. Alexander, T. Biswas and R. H. Brandenberger, arXiv:0707.4679 [hep-th].
- [42] V. F. Mukhanov, Sov. Phys. JETP **67**, 1297 (1988) [Zh. Eksp. Teor. Fiz. **94N7**, 1 (1988)].
- [43] M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986).
- [44] R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, Phys. Rev. D **51**, 6744 (1995), hep-th/9501066.
- [45] D. H. Lyth, Phys. Lett. B **524**, 1 (2002) hep-ph/0106153; D. H. Lyth, Phys. Lett. B **526**, 173 (2002), hep-ph/0110007.
- [46] F. Finelli and R. Brandenberger, Phys. Rev. D **65**, 103522 (2002), hep-th/0112249.
- [47] J. C. Hwang, Phys. Rev. D **65**, 063514 (2002), astro-ph/0109045.
- [48] P. Creminelli, A. Nicolis and M. Zaldarriaga, Phys. Rev. D **71**, 063505 (2005), hep-th/0411270.
- [49] A. J. Tolley, N. Turok and P. J. Steinhardt, Phys. Rev. D **69**, 106005 (2004) [arXiv:hep-th/0306109].
- [50] T. J. Battfeld, S. P. Patil and R. H. Brandenberger, Phys. Rev. D **73**, 086002 (2006) [arXiv:hep-th/0509043].
- [51] P. Creminelli and L. Senatore, arXiv:hep-th/0702165.
- [52] E. I. Buchbinder, J. Khoury and B. A. Ovrut, arXiv:hep-th/0702154; E. I. Buchbinder, J. Khoury and B. A. Ovrut, arXiv:0706.3903 [hep-th].
- [53] A. Notari and A. Riotto, Nucl. Phys. B **644**, 371 (2002) [arXiv:hep-th/0205019].
- [54] F. Finelli, Phys. Lett. B **545**, 1 (2002) [arXiv:hep-th/0206112]; F. Di Marco, F. Finelli and R. Brandenberger, Phys. Rev. D **67**, 063512 (2003) [arXiv:astro-ph/0211276].
- [55] J. L. Lehnert, P. McFadden, N. Turok and P. J. Steinhardt, Phys. Rev. D **76**, 103501 (2007) [arXiv:hep-th/0702153].
- [56] L. R. Abramo and P. Peter, arXiv:0705.2893 [astro-ph].
- [57] P. Peter and N. Pinto-Neto, Phys. Rev. D **66**, 063509 (2002), hep-th/0203013; P. Peter, N. Pinto-Neto and D. A. Gonzalez, JCAP **0312**, 003 (2003), hep-th/0306005; J. Martin and P. Peter, Phys. Rev. Lett. **92**, 061301 (2004), astro-ph/0312488; J. Martin and P. Peter, Phys. Rev. D **68**, 103517 (2003), hep-th/0307077; J. Martin and P. Peter, Phys. Rev. D **69**, 107301 (2004), hep-th/0403173.

- [58] V. Bozza and G. Veneziano, Phys. Lett. B **625**, 177 (2005), hep-th/0502047; V. Bozza and G. Veneziano, JCAP **0509**, 007 (2005), gr-qc/0506040.
- [59] L. E. Allen and D. Wands, Phys. Rev. D **70**, 063515 (2004), astro-ph/0404441.
- [60] F. Finelli, JCAP **0310**, 011 (2003), hep-th/0307068.
- [61] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B **569**, 113 (2003), hep-th/0306113.
- [62] R. H. Brandenberger, publ. in proc. of IPM School On Cosmology 1999: Large Scale Structure Formation, hep-ph/9910410.
- [63] J. Martin and R. H. Brandenberger, Phys. Rev. D **63**, 123501 (2001) [arXiv:hep-th/0005209].
- [64] Y. S. Piao, B. Feng and X. M. Zhang, Phys. Rev. D **69**, 103520 (2004) [arXiv:hep-th/0310206].
- [65] Y. S. Piao, S. Tsujikawa and X. M. Zhang, Class. Quant. Grav. **21**, 4455 (2004) [arXiv:hep-th/0312139].
- [66] Y. S. Piao, Phys. Rev. D **71**, 087301 (2005) [arXiv:astro-ph/0502343].
- [67] I. Y. Aref'eva, L. V. Joukovskaya and S. Y. Vernov, arXiv:hep-th/0701184; L. Joukovskaya, arXiv:0707.1545 [hep-th].
- [68] X. Zhang and Y. Ling, JCAP **0708**, 012 (2007) [arXiv:0705.2656 [gr-qc]].
- [69] H. Wei and S. N. Zhang, Phys. Rev. D **76**, 063005 (2007) [arXiv:0705.4002 [gr-qc]].